Sparsity Structure and Optimality of Multi-Robot Coverage Control

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***Abstract*—The structure of the Hessian matrix obtained from the locational cost used in coverage control is investigated to provide conditions on the optimality of coverage control solutions. It is shown that in arbitrary dimensions, the Hessian matrix is composed of the direct sum of three well-structured matrices: a diagonal matrix, a block-diagonal matrix, and block-Laplacian matrix. This structure is exploited in the one-dimensional case, where an alternative proof of a sufficient condition for optimality is given. A relationship is shown between centroidal Voronoi tessellation (CVT) configurations and the sufficient condition for optimality via the spatial derivative of the density provided in the cost. A decomposition is used to provide insight into the terms which most affect optimality. Several classes of density functions are analyzed under the proposed condition. Experiments on a multi-robot team are shown to verify theoretical results.**

***Index Terms*—Networked control systems, robotics, optimization.**

I. Introduction

With the continuous advance of technology, access to smaller, less expensive, and more capable robots has become more readily available and multi-robot studies have gained in popularity. The study of swarm robotics has had impact in agriculture [1], environmental monitoring [2], human-swarm interaction [3], and others. Coverage control is an application of swarm robotics in which a team of agents is deployed into a known region in an attempt to optimally survey the region. For example, the two-dimensional case is applicable in coverage of planar domains of interest, such as fields or rooms that may be surveyed by ground robots or aerial robots that stay at a constant altitude.

The problem of coverage control using Voronoi tessellations has been studied extensively as in [4], [5], [6], and [7]. In [4], the notion of coverage based on locational optimization and a locational cost was introduced. Locational optimization has been employed in similar problems including quantization, clustering, and the facility location problem. In [5], a decentralized control law is proposed that drives robots to a centroidal Voronoi tessellation (CVT) configuration that is known to be necessary for minimizing the locational cost. In addition to driving robots to a critical configuration, the control law integrates sensor measurements to provide an estimate of the distribution of sensory information in the environment. In [6], the control of a swarm of agents under time-varying density functions was considered and a control law is proposed that guarantees maintaining a CVT configuration over time. In [7], the control of a swarm of robots with heterogeneous sensing capabilities using Voronoi tessellations is investigated via locational optimization.

It is well known that finding the globally optimal configuration given a density function is NP-hard [8], so most work on this problem focuses on locally minimizing configurations. For one-dimensional coverage, originally in [9] and later in [10], it was shown that using a log-concave density function is a sufficient condition for minimizing CVT configurations. Similar conditions for higher dimensions have been elusive, however, using variational techniques, a sufficient condition for a CVT configuration to be a local minimum is given in [11]. In this letter, a new interpretation is given to the Hessian matrix of the locational cost and an alternative proof is given for the one-dimensional sufficient condition, which is found to match the results in [11].

The novelty of this letter is threefold. First, the structure of the Hessian matrix of the locational cost is analyzed and a sparsity structure is exposed. Second, an interpretation is provided based on a decomposition of the rate of change of the center of mass matrix of a Voronoi cell ($\partial c_i/\partial p_j$) and the sparsity structure of the Hessian. Finally, using the sparsity structure and this interpretation, an alternative proof of the main result in [11] is given for cost-minimizing coverage in the one-dimensional case.

One-dimensional coverage has applications in several areas including quantization [10] and guaranteed capture [12]. Recent work, [12], uses one-dimensional coverage under uniform density for guaranteed capture in multi-player dynamic games. While in [12] optimality is observed under the uniform density case, we present more general results for arbitrary densities.

The organization of this letter is as follows: In Section II, some preliminaries for the coverage problem are presented and the locational cost, the metric of optimality, is defined. Section III investigates the Hessian matrix in arbitrary dimensions and demonstrates several formulations for the matrix. In Section IV, we reduce the problem to one dimension, borrowing the new formulation for the Hessian matrix found in prior sections to gain more insight regarding the optimality of the robot configuration, and a sufficient condition is presented for the CVT configuration to minimize the locational cost. Section V verifies the sufficient condition in physical robotic systems, and Section VI provides conclusions.

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II. The Coverage Problem

To discuss minimizing conditions for multi-robot coverage of a domain, a metric must first be defined. We follow the construction of the locational cost as was done in [4].

The locational cost describes how well the configuration of a team of robots covers a desired area of interest (i.e., domain). Define $D \subseteq \mathbb{R}^d$ to be the d-dimensional convex region representing the domain. Within the domain, a team of n robots is deployed, with positions $p_i \in D, i = 1, 2, \ldots, n$. The positions of these robots are collected into a single vector $p = \left[p_1^T, \ldots, p_n^T\right]^T \in \mathbb{R}^{nd}$. To influence the configuration of the robot positions, each point in D is assigned a weight of relative importance. Define $\phi : D \to (0, \infty)$ to be a bounded density function that is continuously differentiable. Then, the relative importance of a point $q \in D$ is captured by $\phi(q)$. Since the performance of a large class of sensors deteriorates at a rate proportional to the square of the distance [13], [14], consider the locational cost

$$\mathcal{H}(p) = \sum_{i=1}^{n} \int_{V_i(p)} \|q - p_i\|^2 \phi(q) dq \tag{1}$$

where the domain D has been partitioned into regions of dominance, $V_i(p)$, where the cost captures how well each robot is covering their own region. While any proper partition of D may be utilized, in this work a Voronoi tessellation is used since it has been shown to minimize the locational cost for a given p with respect to the choice of partition [4]. The Voronoi cell for agent i is then given by

$$V_i(p) = \{q \in D : ||q - p_i|| \leq ||q - p_j||, i \neq j\}. \tag{2}$$

One can define the mass, m_i, and center of mass, c_i, of the ith Voronoi cell, V_i, as the hyper-volume integrals (e.g., area integrals in 2D, volume integrals in 3D)

$$m_i(p) = \int_{V_i(p)} \phi(q) dq \tag{3}$$

$$c_i(p) = \frac{\int_{V_i(p)} q\phi(q) dq}{m_i} \tag{4}$$

where the center of mass of Voronoi cell i is defined to satisfy the vector equality

$$\int_{V_i(p)} (q - c_i)\phi(q) dq = 0_d = [0, \ldots, 0]^T \in \mathbb{R}^d.$$

In [10] and [15], it was shown that the first derivative of the locational cost is given by

$$\frac{\partial \mathcal{H}}{\partial p_i} = \int_{V_i(p)} -2(q - p_i)^T \phi(q) dq = 2m_i(p_i - c_i)^T. \tag{5}$$

Since $\phi > 0$, from (5), a critical point of (1) is given by

$$p_i = c_i(p), \ i = 1, \ldots, n \tag{6}$$

or equivalently

$$\int_{V_i(p)} (q - p_i)\phi(q) dq = 0_d \qquad \forall i. \tag{7}$$

Thus, a minimizer to (1) is necessarily of this form. However, when (6) is satisfied, p is said to be a centroidal Voronoi tessellation (CVT) [16]. We are interested in determining conditions under which a CVT is a minimizing configuration with respect to the locational cost. To do this, the Hessian matrix at a CVT configuration is investigated.

III. The Hessian Matrix

In this section we explore the structure of the Hessian Matrix to the locational cost. This structure will be used to provide conditions for the matrix to be positive semi-definite at a CVT, yielding a local minimum of the locational cost. Define $c = \left[c_1^T, \ldots, c_n^T\right]^T \in \mathbb{R}^{nd}$ to be the vector of center of masses. By direct computation, it can be shown that

$$\frac{\partial^2 \mathcal{H}}{\partial p_i \partial p_j} = 2\left(\frac{\partial m_i}{\partial p_j}(p_i - c_i)^T + m_i\left(\frac{\partial p_i}{\partial p_j} - \frac{\partial c_i}{\partial p_j}\right)\right).$$

Since only CVT configurations are considered, we have that $(p_i - c_i) = 0_d$. Additionally, $\partial p_i / \partial p_j = 0_{d\times d}$ for $i \neq j$, and is identity otherwise. If $M \in \mathbb{R}^{nd\times nd}$ is defined to be a block-diagonal matrix where each diagonal block is a $d \times d$ matrix, given by $[M]_{ii} = m_i I_d$ where I_d is the d-dimensional identity matrix, then the Hessian matrix may be expressed as

$$\left.\frac{\partial^2 \mathcal{H}}{\partial p^2}\right|_{p=c} = 2M\left(I_{nd} - \frac{\partial c}{\partial p}\right). \tag{8}$$

A. Sparsity Structure

In [6] it was shown that $\left[\frac{\partial c}{\partial p}\right]_{ij} = \frac{\partial c_i}{\partial p_j} \in \mathbb{R}^{d\times d}$ is given by

$$\frac{\partial c_i}{\partial p_j} = \int_{\partial V_{ij}} \frac{(q - c_i)(p_j - q)^T}{m_i ||p_j - p_i||} \phi(q) dq \tag{9}$$

and

$$\frac{\partial c_i}{\partial p_i} = -\sum_{k\in N_{V_i}} \int_{\partial V_{ik}} \frac{(q - c_i)(p_i - q)^T}{m_i ||p_k - p_i||} \phi(q) dq \tag{10}$$

where $\partial V_{ij} = V_i \cap V_j$ is the set of points shared by Voronoi cells i and j. N_{V_i} is agent i's neighborhood set for the Delaunay graph induced by the Voronoi tessellation. It is important to note that these integrals are $(d - 1)$-dimensional integrals rather than d-dimensional integrals (e.g., line integrals if the domain $D \subseteq \mathbb{R}^2$). If agents i and j are not neighbors with respect to the Delaunay graph, $\partial c_i / \partial p_j = 0_{d\times d}$. Thus, $\partial c / \partial p$ encodes adjacency information for the induced Delaunay graph.

As investigated in [16], the hyperplane between two Voronoi cells is orthogonal to the line connecting the Voronoi cell generator points. Due to this special geometry, we now use a fact from specular reflection to relate $(p_i - q)$ to $(p_j - q)$ for $q \in \partial V_{ij}$. Let the unit vector normal to the boundary between Voronoi cells i and j be defined as $n_{ij} = (p_j - p_i)/||p_j - p_i||$ (outward from cell i) and define the boundary-tangent unit vector to be given by t_{ij}, orthogonal to n_{ij}, i.e., such that $n_{ij}^T t_{ij} = 0$. Then if we define $N_{ij} = n_{ij} n_{ij}^T$ to be the normal projection matrix and $T_{ij} = t_{ij} t_{ij}^T = I_d - N_{ij}$ to be orthogonal projection matrix, there exists a dyadic relation $N_{ij} + T_{ij} = I_d$. Note that if $d = 2$, we may simply define $t_{ij} = S n_{ij}$ where

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

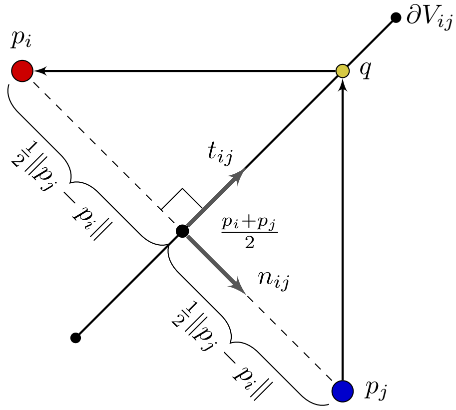

Fig. 1. Two-dimensional specular reflection example. The vector $(q - p_j)$ is in the incident direction, and $(p_i - q)$ is in the specularly reflected direction.

Furthermore, for $d = 1$, we have $n_{ij} = \pm 1$ and $t_{ij} = 0$.

Let $(p_i - q)$ be the specularly reflected direction and $(q - p_j)$ be the incident direction (see Fig. 1). From [17], we have

$$p_i - q = \left(I - 2t_{ij}t_{ij}^T\right)(q - p_j) = (2T_{ij} - I)(p_j - q). \quad (11)$$

By substituting this relation into our expression for $\partial c_i/\partial p_i$, one may verify that

$$\frac{\partial c_i}{\partial p_i} = \sum_{k \in N_{V_i}} \frac{\partial c_i}{\partial p_k}(2N_{ik} - I).$$

From here, define a weighted block-Laplacian matrix [18], given by $\mathcal{L} = \Delta - A$ where A is the weighted block-adjacency matrix with $[A]_{ij} = m_i \partial c_i/\partial p_j$ when $i \neq j$ and zero when $i = j$. Δ is the degree block-diagonal matrix with $[\Delta]_{ii} = \sum_{k \in N_{V_i}} A_{ik}$. Further, define the block-diagonal matrix $\mathcal{D}$ by $[\mathcal{D}]_{ii} = \sum_{k \in N_{V_i}} 2A_{ik}N_{ik}$. Using these, we find an alternate formulation for the Hessian matrix.

Lemma 3.1: At a CVT configuration, the Hessian matrix may be expressed as

$$\left.\frac{\partial^2 \mathcal{H}}{\partial p^2}\right|_{p=c} = 2(M + \mathcal{L} - \mathcal{D}). \quad (12)$$

Proof: Follows from direct substitution. ■

Each of the terms in the Hessian matrix is well-structured.

B. Matrix Decomposition of $\partial c/\partial p$

Consider the matrix $m_i \partial c_i/\partial p_j$. Define $\Delta c_i = (p_i - c_i)$ to be the deviation from a CVT for agent i. Additionally define $b(q) = (q - p_i)^T t_{ij}$ and further let

$$u = \tfrac{1}{2}||p_j - p_i||n_{ij}, \qquad \delta_n = N_{ij}\Delta c_i$$
$$v = T_{ij}(q - p_i) = b(q)t_{ij}, \quad \delta_t = T_{ij}\Delta c_i.$$

Then u and v are the normal and orthogonal components of $(q - p_i)$ and $(p_j - q)$, as in Fig. 1. Similarly, δ_n and δ_t are the normal and orthogonal components of Δc_i. Then $m_i \partial c_i/\partial p_j$ may be expressed as

$$m_i \frac{\partial c_i}{\partial p_j} = \int_{\partial V_{ij}} \frac{(u + v + \delta_n + \delta_t)(u - v)^T}{||p_j - p_i||} \phi(q) dq.$$

At a CVT configuration, δ_n and δ_t both become zeroes. If the outer product expansion is carried out at a CVT configuration and one defines $\mu_{ij}(p) := \int_{\partial V_{ij}} \phi(q) dq$ to be the mass along the boundary shared by Voronoi cells i and j, we find that

$$m_i \frac{\partial c_i}{\partial p_j} = \frac{1}{4}||p_j - p_i||\mu_{ij}\, N_{ij} - \int_{\partial V_{ij}} \frac{b^2(q)\phi(q)}{||p_j - p_i||} T_{ij}\, dq - \frac{1}{2}\int_{\partial V_{ij}} b(q)\phi(q)\, S_{ij}\, dq$$

where the matrix $S_{ij} = Q_{ij} - Q_{ij}^T$ is skew-symmetric, with $Q_{ij} = n_{ij}t_{ij}^T$. This alternate representation provides a natural decomposition of the blocks in the matrix $\partial c/\partial p$: two terms (with N_{ij} and T_{ij}) account for components longitudinal and transversal to the normal (respectively), and the third term (with S_{ij}) may be interpreted as accounting for infinitesimal rotations orthogonal to these two.

Now consider the $\mathcal{D}$ matrix. By the previous expression, we may verify that

$$\begin{aligned}\mathcal{D}|_{p=c} &= 2m_i \sum_{j \in N_{V_i}} \frac{\partial c_i}{\partial p_j} N_{ij} \\ &= \sum_{j \in N_{V_i}} \left(\tfrac{\|p_j - p_i\|}{2}\mu_{ij}N_{ij} + \int_{\partial V_{ij}} b(q)\phi(q)\, Q_{ij}^T\, dq\right)\end{aligned} \quad (13)$$

the interpretation being that at a CVT, the block entries in $\mathcal{D}$ are only affected by part of the infinitesimal rotation and the longitudinal components of each $\partial c_i/\partial p_j$.

IV. The One-Dimensional Case

In previous sections, the coverage problem was addressed for arbitrary dimensions. We now consider the case where $D \subseteq \mathbb{R}$, and provide a sufficient condition for the optimality of a CVT configuration. When $d = 1$, each of the $\partial c_i/\partial p_j$ blocks become scalar, $N_{ij} = 1$, and $T_{ij} = S_{ij} = Q_{ij} = 0$ by virtue of $n_{ij} = \pm 1$ and $t_{ij} = 0$. By direct substitution, we note that $[\mathcal{L} - \mathcal{D}]_{ii} = -\sum_{k \in N_{V_i}} m_i \partial c_i/\partial p_j$ and $[\mathcal{L} - \mathcal{D}]_{ij} = -m_i \partial c_i/\partial p_j$. Thus, we can express $\mathcal{L} - \mathcal{D} = -(\Delta + A)$. That is, in one dimension, the Hessian may be further expressed as

$$\frac{\partial^2 \mathcal{H}}{\partial p^2} = 2(M - \Delta - A).$$

Without loss of generality we may order the agents such that $p_i < p_j$ if $i < j$. When $d = 1$, the Delaunay graph is a line graph, such that an agent has two neighbors if it is in the interior of graph, and only one if it is at the boundary. Assume agent i lies in the interior and has two neighbors, and denote the two neighbors to be h and j such that $p_h < p_i < p_j$. Then the boundary between cells V_i and V_j (or V_h) is a singleton, given by $\partial V_{ij} = (p_i + p_j)/2$ (or $\partial V_{ih} = (p_i + p_h)/2$).

Note that the matrix M is now given by $M = \text{diag}([m_1, m_2, \ldots, m_n])$. To compute $\partial c_i/\partial p_j$ via (9), we must interpret the integration operation as a zero-dimensional integral over a zero-dimensional manifold. As the unit-normal outward direction has already been taken into account in the definition of $\partial c_i/\partial p_j$ [6], the integration amounts to summing the integrand evaluates at the points in the domain of integration. Thus, the partial derivative evaluates to

$$\begin{aligned}\frac{\partial c_i}{\partial p_j} &= \int_{\{\partial V_{ij}\}} \frac{(q-p_i)(p_j-q)}{m_i||p_j-p_i||}\phi(q)dq \\ &= \frac{\left(\frac{p_i+p_j}{2}-p_i\right)\left(p_j-\frac{p_i+p_j}{2}\right)}{m_i||p_j-p_i||}\phi\left(\frac{p_i+p_j}{2}\right) \\ &= \frac{1}{4}\frac{p_j-p_i}{m_i}\phi\left(\frac{p_i+p_j}{2}\right).\end{aligned}$$

Similarly,

$$\frac{\partial c_i}{\partial p_h} = \frac{1}{4}\frac{p_i-p_h}{m_i}\phi\left(\frac{p_i+p_h}{2}\right).$$

Note this is exactly what is obtained from (13) since the second term disappears in one dimension.

A. A Sufficient Condition for Optimality

We now use these expressions to compute the entries of the Hessian matrix. For the off-diagonal elements we have

$$-m_i\frac{\partial c_i}{\partial p_j} = -\frac{1}{4}(p_j-p_i)\phi\left(\frac{p_i+p_j}{2}\right) = \frac{1}{2}\left[\frac{\partial^2\mathcal{H}}{\partial p^2}\right]_{ij}$$

and for the diagonal elements

$$\left[\frac{\partial^2\mathcal{H}}{\partial p^2}\right]_{ii} = 2m_i - \frac{(p_j-p_i)}{2}\phi\left(\frac{p_i+p_j}{2}\right) - \frac{(p_i-p_h)}{2}\phi\left(\frac{p_i+p_h}{2}\right).$$

From here, it is simple to check that if we substitute $\phi = 1$ and note that at the CVT $m_i = p_j - p_i = p_i - p_h$, these results match those in [10].

For the two agents on the boundaries, 1 and n, as they only have one neighbor we find that

$$\begin{aligned}\left[\frac{\partial^2\mathcal{H}}{\partial p^2}\right]_{11} &= 2m_1 - \frac{1}{2}(p_2-p_1)\phi\left(\frac{p_1+p_2}{2}\right) \\ \left[\frac{\partial^2\mathcal{H}}{\partial p^2}\right]_{nn} &= 2m_n - \frac{1}{2}(p_n-p_{n-1})\phi\left(\frac{p_{n-1}+p_n}{2}\right).\end{aligned}$$

Lemma 4.1: The Hessian matrix $\partial^2 H/\partial p^2|_{p=c}$ is positive semi-definite if for all agents i with two neighbors we have

$$m_i \geq \left(\frac{(p_j-p_i)}{2}\phi\left(\frac{p_i+p_j}{2}\right) + \frac{(p_i-p_h)}{2}\phi\left(\frac{p_i+p_h}{2}\right)\right)$$

and for agent 1

$$m_1 \geq \frac{1}{2}(p_2-p_1)\phi\left(\frac{p_1+p_2}{2}\right)$$

and for agent n

$$m_n \geq \frac{1}{2}(p_n-p_{n-1})\phi\left(\frac{p_n+p_{n-1}}{2}\right).$$

Proof: This result follows directly from an application of Gershgorin's circle theorem [19] on the rows of the Hessian matrix. Since the matrix is diagonally dominant, all eigenvalues are greater than or equal to zero, so the Hessian is positive semi-definite. ■

With this lemma we can now provide a sufficient condition on cost-minimizing coverage in one-dimension, captured in the following theorem.

Theorem 4.2: The matrix

$$\frac{\partial^2\mathcal{H}}{\partial p^2}\Big|_{p=c} = 2M(I - \frac{\partial c}{\partial p})$$

is positive semi-definite if

$$\int_{V_i(p)} \frac{d\phi}{dq}(q-p_i)dq \leq 0 \;\; \forall i. \tag{14}$$

Proof: First, consider an agent i with two neighbors. Let $V_i(p) = [\partial V_{ih}, \partial V_{ij}]$. By linearity, we may break the integral into two terms

$$\int_{V_i(p)} \frac{d\phi}{dq}(q-p_i)dq = \int_{V_i(p)} \frac{d\phi}{dq} q\, dq - p_i \int_{V_i(p)} \frac{d\phi}{dq}\, dq.$$

From the fundamental theorem of calculus and an application of integration by parts, we expand the integrals to get

$$\begin{aligned}&= \partial V_{ij}\phi(\partial V_{ij}) - \partial V_{ih}\phi(\partial V_{ih}) - \int_{V_i(p)}\phi(q)dq \\ &\qquad - p_i(\phi(\partial V_{ij}) - \phi(\partial V_{ih})).\end{aligned}$$

The sole integral term remaining is $\int_{V_i(p)}\phi(q)dq = m_i(p)$. Assuming, without loss of generality, that $p_h < p_i < p_j$ and substituting in the value of the boundary points, the above expression becomes

$$\begin{aligned}\int_{V_i(p)} \frac{d\phi}{dq}(q-p_i)\, dq &= \frac{p_j-p_i}{2}\phi\left(\frac{p_i+p_j}{2}\right) \\ &+ \frac{p_i-p_h}{2}\phi\left(\frac{p_i+p_h}{2}\right) - m_i(p) \leq 0.\end{aligned}$$

Solving for $m_i(p)$ yields

$$m_i(p) \geq \frac{p_j-p_i}{2}\phi\left(\frac{p_i+p_j}{2}\right) + \frac{p_i-p_h}{2}\phi\left(\frac{p_i+p_h}{2}\right)$$

which is the same inequality obtained in Lemma 4.1.

We now show this for agent 1. Note that the Voronoi cell for agent 1 is $[a, (p_1+p_2)/2]$, where a is the lower bound of the domain. Expanding the integral in a similar manner yields

$$m_1(p) \geq \frac{1}{2}\left((p_2-p_1)\phi\left(\frac{p_1+p_2}{2}\right) + (p_1-a)\phi(a)\right).$$

Here, since $\phi(a) > 0$ and $p_1 \geq a$, we certainly have that

$$m_1(p) \geq \frac{1}{2}(p_2-p_1)\phi\left(\frac{p_1+p_2}{2}\right)$$

which is the same as the inequality from Lemma 4.1. A similar argument holds for agent n. ■

Remark 1: We make a special note of the similarity between the necessary condition for optimality in (7) and the sufficient condition in (14).

In [11], variational methods were utilized to get a sufficient condition in the arbitrary dimension case. If we define

$$\frac{\partial\phi}{\partial q} = \left[\frac{\partial\phi}{\partial q_1}, \ldots, \frac{\partial\phi}{\partial q_d}\right],$$

the sufficient condition for optimal coverage in d-dimensions is given by

$$\begin{aligned}\int_{V_i(p)} \frac{\partial\phi}{\partial q}(p_i-q)\, dq &\geq (d-1)m_i \\ &+ 2\sum_{j\in N_{V_i}} \int_{\partial V_{ij}} \frac{b^2(q)}{||p_j-p_i||}\phi(q)dq, \;\; \forall i.\end{aligned} \tag{15}$$

That is, if (15) holds for all agents, the Hessian matrix is positive semi-definite, and if strict inequality holds for at least one, then the matrix is positive definite. Note that for $d = 1$, this immediately reduces to the sufficient condition given in (14), since $b(q) = 0$ in the one-dimensional case.

Remark 2: Note the additional terms on the right-hand side of (15). The T_{ij} term from $m_i \partial c_i / \partial p_j$ appears in the general sufficient condition, while the N_{ij} and S_{ij} terms do not. Thus, one take away from the decomposition is that the T_{ij} component could be maximized to improve the likelihood of optimal coverage at a CVT configuration.

We consider the application of (14) to the construction of density functions that guarantee convergence to cost-minimizing CVT. In the next subsection, the condition in Theorem 4.2 is applied to certain classes of density functions.

B. Application to Classes of Density Functions

First, consider the case where the equality holds. Trivially, the case where $d\phi/dq = 0 \ \forall q$ maintains the equality. That is, any constant density function will satisfy (14). In [12], an argument is given to prove that the matrix $I - \partial c/\partial p$ is invertible under a uniform density. Since in the uniform density case at a CVT configuration all agents have the same mass $m = m_i \ \forall i$, the Hessian matrix $\partial^2 \mathcal{H}/\partial p^2 = 2M(I - \partial c/\partial p) = 2m(I - \partial c/\partial p)$ is positive definite, and the CVT configuration minimizes the locational cost.

Clearly, if $d\phi/dq = a\phi(q)$, then the equality must hold by (7). Solutions to this differential equation are of the form $\phi = \phi_0 \exp(a(q - q_0))$. So an exponential density function also satisfies (14) implying that at any CVT configuration the Hessian is positive semi-definite.

Consider an arbitrary linear density function $\phi(q) = aq + b$ where $a, b \in \mathbb{R}$, $a \neq 0$, and $\phi > 0$ for all $q \in D$. A direct substitution into (14) yields the condition

$$p_i \geq \tfrac{1}{2}(p_j - p_h)$$

for $a > 0$ and the opposite inequality for $a < 0$. This is what we would expect a CVT configuration to look like for a monotonic density function and this is, in fact, a necessary condition for $p_i = c_i$ under monotonic ϕ. So, as (14) is satisfied strictly for a CVT configuration under a linear density function, the configuration always minimizes (1).

One density that does not satisfy (14) is the function $\phi(q) = a\exp(bq) + d$ for $a, b, d \in \mathbb{R}_+$. Noting $d\phi/dq = b\phi - bd$, substituting into (14) and using $p_i = c_i$ yields

$$\int_{V_i} (q - c_i) \tfrac{d\phi}{dq} dq = b \int_{V_i} (q - c_i)\phi(q) dq - bd \int_{V_i} (q - c_i) dq.$$

However, the first term on the right-hand side goes to zero by (7). If $|V_i|$ is defined to be the length of Voronoi cell i and $\overline{V_i}$ to be the midpoint of Voronoi cell i, then

$$\begin{aligned}\int_{V_i} (q - c_i) \tfrac{d\phi}{dq} dq &= -bd \int_{V_i} q dq + bdc_i |V_i| \\ &= -bd|V_i|\overline{V_i} + bdc_i|V_i| = -bd|V_i|(\overline{V_i} - c_i).\end{aligned}$$

For ϕ to satisfy (14) we would need for $-bd|V_i|(\overline{V_i} - c_i) \leq 0$, but this is only the case if $\overline{V_i} \geq c_i$. Since ϕ is monotonic this is not possible, so (14) is not satisfied.

V. Experimental Results

The results from previous sections were tested on a multi-robot team at the Robotarium [20], a remotely accessible swarm robotics testbed at the Georgia Institute of Technology. The team is composed of six GRITSBots [21], which are differential-drive mobile robots. A webcam-based tracking system provides information about the positions and orientations of the robots. This data is fed into the control law, which provides the velocity commands to the robots.

We are interested in verifying that the sufficient condition in (14) would allow us to ascertain the optimality of the final configuration of a multi-robot team that is driven from a random initial configuration to achieve a CVT configuration, noting that this final configuration might be close but not exactly at a CVT due to disturbances and unmodeled phenomena. We use the TVD-C algorithm that was proposed in [6] to drive the robots to a CVT configuration. The TVD-C control law is given by

$$\dot{p} = \left(I - \tfrac{\partial c}{\partial p}\right)^{-1} \left(-\kappa(p - c) + \tfrac{\partial c}{\partial t}\right) \qquad (16)$$

where $\kappa > 0$ is a tuning parameter. Note that since we only consider static densities, $\partial c/\partial t = 0$. To test the result of coverage in one-dimension, we initially drive the robots to initial positions on a line, and then orient them so they all face the same direction. From there, one-dimensional coverage begins. The final configuration achieved is shown in Fig. 2

Seven different density functions were tested: uniform, linear, exponential, exponential with constant offset, Gaussian, bimodal density, and a sine cubed density. All functions are defined over domain $[0, 1]$ and codomain $[1/6, 1]$. For convenience, we label the density functions $\phi_1(q), \phi_2(q), \ldots, \phi_7(q)$, respectively, defined as follows

$$\begin{aligned}
&\phi_1(q) = 1, \qquad\qquad \phi_2(q) = \tfrac{a}{b}(1 - q) + q \\
&\phi_3(q) = \tfrac{a}{b}\exp\left(q \log\left(0.9\tfrac{b}{a}\right)\right), \quad \phi_4(q) = \phi_3(q) + 0.1 \\
&\phi_5(q) = \exp\left[-(q - 0.5)^2\left(-4\log\left(\tfrac{a}{b}\right)\right)\right] \\
&\phi_6(q) = d\big[\exp(\tfrac{-(q-0.25)^2}{c}) + \exp(\tfrac{-(q-0.75)^2}{c})\big] \\
&\phi_7(q) = \tfrac{(b-a)\cos^3(2N\pi(q-0.5)+\pi)+(a+b)}{2b}
\end{aligned}$$

where in our experiments $N = 6$ is the number of agents, and

$$a = 0.2,\ b = 1.2,\ c = \tfrac{1}{16\log(b/a)},\ d = (\tfrac{b}{a\exp(-1)})^4.$$

For each density function, the robots were given initial positions $p_0 = [0.25, 0.53, 0.60, 0.67, 0.75, 0.90]^T$, at which point they began coverage via (16) with $\kappa \in [0.5, 2]$.

TABLE I
Maximum value for the sufficient condition and minimum eigenvalue of the Hessian matrix at a CVT

Density	Log-concave	Maximum value for (14)	$\lambda_{\min}$
ϕ_1	✓	0	0.581
ϕ_2	✓	-1.24×10^{-3}	0.345
ϕ_3	✓	-4.12×10^{-4}	0.237
ϕ_4		4.74×10^{-3}	0.313
ϕ_5	✓	-1.12×10^{-3}	0.335
ϕ_6		8.52×10^{-2}	0.506
ϕ_7		-6.46×10^{-1}	0.520

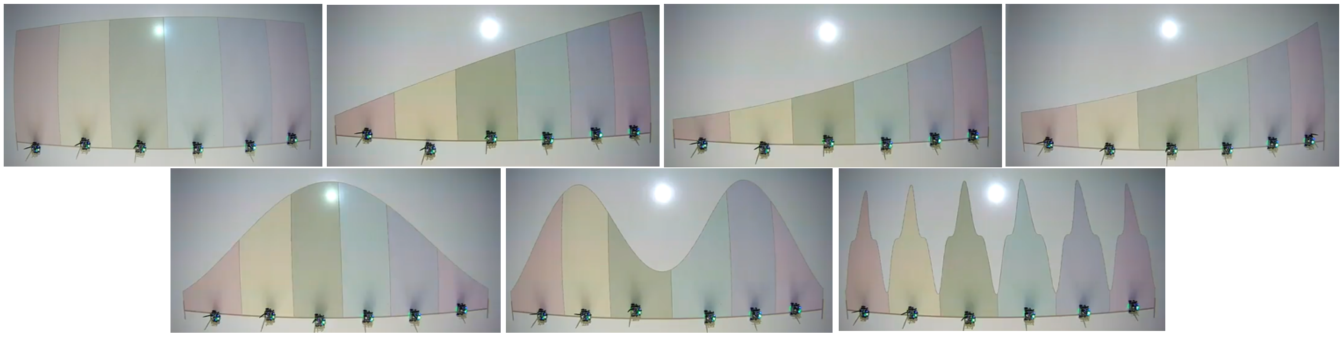

Fig. 2. The final centroidal Voronoi tessellation configurations achieved by six robots performing 1D coverage in the Georgia Tech Robotarium under seven different static density functions. In the top row, from left to right: uniform, linear, exponential, and exponential with offset densities. In the bottom row from left to right: Gaussian, bimodal, and sine cubed densities. For the different densities, note the different spacing in the tessellations. An overhead projector is used to visualize several elements in real time directly on the robot workspace. In the plots: the 1D domain is the x-axis (towards the bottom where the robots are lined up), above it the 1D density is plotted. The density is partitioned based on the Voronoi tessellation of the domain, where the area under the curve for each partition is colored differently. For a video see: https://youtu.be/Zpz-Co44Zyg

Once the agents arrived at a CVT configuration, the minimum eigenvalue, $\lambda_{\min}$, of the Hessian matrix was collected for each density and as well as the maximum values of (14) with respect to each agent, i.e.,

$$\max\{\int_{V_i(p)} (q - p_i)\frac{d\phi}{dq} dq : i = 1, \ldots, 6\}.$$

These values are presented in Table I. We may see that the sufficient condition was satisfied for all log-concave densities, but it is also satisfied by ϕ_7, which is not log-concave.

Although (14) is not satisfied for all densities, all the final configurations minimize the locational cost since all the eigenvalues of the Hessian matrix are positive.

VI. Conclusions

The optimality of coverage control configurations of multi-robot teams over arbitrary dimensional domains is considered, where a fact from specular reflection exposes a Laplacian-like structure in the Hessian matrix of the locational cost. This structure is exploited in the one-dimensional case to obtain a sufficient condition on cost-minimizing coverage configurations. The condition is applied to different classes of density functions and it demonstrated that certain classes of density always result in cost-minimizing coverage. The results are validated in robotic implementation where these density functions are executed in experiments where robot agents optimally cover the domain, and it is shown that these density functions provide cost-minimizing coverage.

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