Pursuer Coordination in Multi-Player Reach-Avoid Games through Control Barrier Functions

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***Abstract*—This letter addresses the problem of coordinating a team of pursuers to capture a fast evader in reach-avoid games. Coordination is achieved through the minimization of a locational cost objective, which results in a coverage control strategy. Coverage control dynamics are equivalently expressed as consensus dynamics. This formulation is used to guarantee optimal coverage of an implicit defense manifold without needing to rely on projection mappings as in previous works. Control barrier functions (CBFs) are used to guarantee the existence of a pursuer-defendable defense manifold for as long as feasible throughout the engagement and guarantee capture. A greedy evader strategy is proposed in terms of the solution to a quadratically constrained quadratic program, and simulation results are presented comparing the performance of an uncoordinated strategy and a pure coverage strategy to the proposed coordinated CBF strategy.**

***Index Terms*—Cooperative control, decentralized control, networked control systems.**

I. Introduction

REACH-AVOID (RA) games are a type of differential game in which one player, the evader, attempts to reach a target set while actively avoiding other sets, typically induced by the remaining players. RA games have many applications including safety in robotics, surveillance, and defense. Optimal strategies in these games require solving the computationally expensive Hamilton-Jacobi-Isaacs equation, but suboptimal strategies have also been considered in [1] and [2].

Most work on RA games has been focused on problems where the pursuers are faster than the evader as in [3] and [4], but games where the evader is faster have practical applications as in the so-called "cops and robber game" [5]. However, cops and robber games have been primarily studied on graphs, which do not capture the vehicle trajectory planning aspect of the problem addressed in this work.

This letter expands on the existing strategies provided in [2]. While pursuer coordination via coverage control is generalized to agents with nonlinear dynamics in arbitrary dimensions in [2], the strategies do not provide any guarantees on being able to maintain the sufficient condition for capture. In [2], the pursuers coordinate their coverage efforts over the defense manifold constructed by the intersection of Apollonius circles. However, in this letter, the problems of coordination and defense manifold maintenance are coupled, where the defense manifold becomes an implicitly defined manifold arising from performing coverage directly on the Apollonius circles.

Although in some games it is impossible to maintain a pursuer-defendable defense manifold for all times throughout the engagement, we propose a mechanization to guarantee the maintenance for as long as it is feasible through the use of control barrier functions (CBFs) for set forward invariance [6]. While CBFs have been used in the past in cooperative problems such as multirobot collision avoidance [7], safety in bipedal robotic walking [6], and persistent robotic coverage [8], to the best of our knowledge, they have not been used in adversarial settings such as those in RA games.

The contributions of this letter are thus threefold. First, the requirement of explicitly constructing a defense manifold and an appropriate mapping of the coverage dynamics has been relaxed; instead, coverage is performed over an implicitly defined manifold in a decentralized way. Secondly, robustness guarantees of the dynamics are provided in terms of the rate of change of the defense manifold and the heterogeneity of the pursuers. Lastly, through the use of CBFs, we ensure a pursuer-defendable defense manifold is maintained in a decentralized way throughout the engagement for as long as is feasible while still making progress toward capturing the evader.

The organization of this letter is as follows: Section II provides definitions for the RA game under consideration. Section III gives definitions and assumptions on the constrained reachable set (CRS) and the defense manifold. Section IV discusses coverage control over one-dimensional manifolds and provides an equivalence between coverage control on these manifolds and consensus dynamics in leader-follower networks. Section V defines time-varying CBFs and proposes a decentralized strategy for pursuers to guarantee the existence of a pursuer-defendable defense manifold while coordinating their coverage throughout the game. Section VI provides experimental results against a fast evader using its own CBF to avoid capture. Finally, Section VII provides conclusions.

II. RA Game Definition

This letter considers the coordination of a team of N pursuers attempting to capture one faster evader for a game with a finite final time, t_f. The evader tries to reach a goal set, $\mathcal{P}$, while actively avoiding capture by the pursuer team before the final time. The finite final time is used to encode the finite energy budget of the evader. Ensuring t_f is finite guarantees the existence of bounded domains over which the

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pursuers may coordinate their efforts. Throughout this letter, the superscript (e) denotes the evader and (i) the i^{th} member of the pursuer team. Further, define $[N] := \{1, \ldots, N\}$. In this work, it is assumed the players have dynamics

$$\dot{x}^e = u^e, \qquad \dot{x}^i = u^i, \tag{1}$$

for states $x^e, x^i \in \mathbb{R}^2$ and inputs $u^e \in \mathcal{U}^e := \{u \in \mathbb{R}^2 \mid \|u\| \leq \overline{u}^e\}, u^i \in \mathcal{U}^i := \{u \in \mathbb{R}^2 \mid \|u\| \leq \overline{u}^i\}$, where $\overline{u}^e, \overline{u}^i$ are the corresponding maximum speeds.

We address the same problem as in [2] and reproduce it along with several important definitions and results:

Problem 1 (RA Game in Finite Time). The RA game in finite time is defined as a game of a kind, in which the evader only wins if it reaches the desired target set, $\mathcal{P}$, before the predefined final time, t_f. If the evader is captured by any of the pursuers or does not reach $\mathcal{P}$ by t_f, the pursuers win. Pursuer i is said to ε-capture the evader at time $t \in [t_0, t_f]$ if $\|x^e(t) - x^i(t)\| \leq \varepsilon$ for a fixed $\varepsilon > 0$.

Lemma 2.1 (Bounded Game): Assume an RA game in finite time and players with dynamics as in (1). Then the game always evolves on a bounded domain for the states.

Definition 1 (Constrained Reachable Set). Define $\mathcal{X}_0$ as a feasible set of states at time t_0 and $\mathcal{X}_f$ as the desired set of states at time t_f. Sets $\mathcal{X}_0$ and $\mathcal{X}_f$ are used to describe feasible initial and final state sets for the evader, which could be used to encode uncertainty and target set objectives, respectively. The constrained reachable set (CRS) is defined as

$$\begin{aligned}\mathcal{R}^e_c(t_0, \mathcal{X}_0, \mathcal{X}_f) = \{x \in \mathbb{R}^2 \mid \dot{x} = u(x,t), x(t_0) \in \mathcal{X}_0, \\ x(t_f) \in \mathcal{X}_f, u(t) \in \mathcal{U}^e, t \in [t_0, t_f]\}.\end{aligned} \tag{2}$$

Definition 2 (Pursuer-Defendable Defense Manifold). A defense manifold, $\mathcal{M}(t)$, is defined as a zero measure set that properly partitions the CRS into two sets, $\mathcal{D}_{\{1,2\}}$, and separates the evader and $\mathcal{P}$. That is, $\mathcal{D}_1 \cup \mathcal{D}_2 = \mathcal{R}^e_c(t, x^e(t), \mathcal{P})$, $\mathcal{D}_1 \cap \mathcal{D}_2 = \mathcal{M}(t)$, $x^e(t) \in \mathcal{D}_1$, and $\mathcal{P} \subseteq \mathcal{D}_2$. $\mathcal{M}(t)$ is said to be pursuer-defendable if $\forall m \in \mathcal{M}(t), \exists i \in [N]$ such that x^i can reach m before the evader can.

Theorem 2.2: A pursuer coordination strategy that continually reconfigures the pursuer team such that $\forall t \in [t_m, t_f] \subseteq [t_0, t_f]$, there exists a pursuer-defendable $\mathcal{M}(t)$, provides a pursuer strategy to solve Problem 1.

Two centralized strategies for coordination are provided in [2] based on coverage control over explicitly constructed defense manifolds. A similar approach is taken in this letter, but a distributed coverage control law is proposed based on a leader-follower consensus formulation.

III. Defense Manifold and Constrained Reachable Set Definition

A. Constrained Reachable Set

The CRS, $\mathcal{R}^e_c(t, x^e(t), \mathcal{P})$, captures the set of all reachable positions of the evader while still reaching the goal set, $\mathcal{P}$, before t_f. The CRS provides a natural way to bound the RA game with respect to the dynamics of the evader. For pursuers to coordinate their efforts via coverage, a coverage domain must be defined. The CRS provides the boundary of this

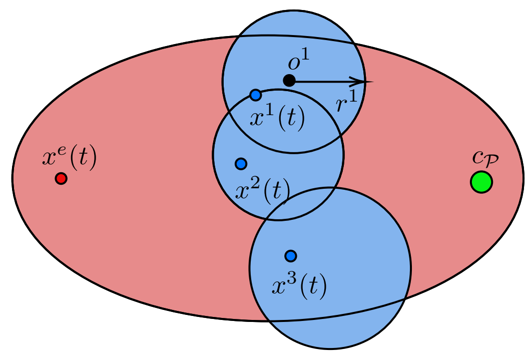

Fig. 1: The CRS is shown by the light red ellipse, whereas Apollonius circles are shown in light blue. Because $\mathcal{Y}^p$ is connected and the boundary Apollonius circles intersect the boundary of the CRS, there exists a pursuer-defendable defense manifold.

domain such that the pursuers are only required to coordinate within the set of reachable positions by the evader.

The CRS on a finite time interval is a compact set, [2]. For the dynamics given in (1), the CRS is given by

$$\begin{aligned}\mathcal{R}^e_c(t_0, x^e(t_0), \mathcal{P}) = \cup_{q \in \mathcal{P}}\{z \in \mathbb{R}^2 \mid \|z - x^e(t_0)\| \\ + \|z - q\| \leq (t_f - t_0)\overline{u}^e\}.\end{aligned}$$

Intuitively, this is the union of ellipses where the foci are the evader's starting position and a point in the target set, $\mathcal{P}$. If the initial starting position is uncertain but is known to be contained within some initial set, $\mathcal{X}_0$, an additional union over all points in $\mathcal{X}_0$ recovers the CRS.

Remark 1: In this work, we assume $\mathcal{P}$ is a singular point, $c_\mathcal{P} \in \mathbb{R}^2$. In general, this is not a limiting assumption because if $\mathcal{P}$ is a compact, convex set, one can compute the centroid, $c_\mathcal{P}$, of $\mathcal{P}$ and consider the largest ellipse with foci $x^e(t_0)$ and $c_\mathcal{P}$ containing the CRS as defined previously. This ellipse will define the boundary of a defense manifold.

B. Pursuer-Defendable Defense Manifold

Define the kinematic disadvantage of pursuer i to the evader to be $\sigma^i := \overline{u}^i / \overline{u}^e < 1$. In this work, the set of points reachable by pursuer i before the evader is approximated using an Apollonius circle [9]. The Apollonius circle encloses the pursuer with radius and center given by

$$r^i = \|x^i - x^e\| \sigma^i c_{\sigma^i}, \qquad o^i = [x^i - (\sigma^i)^2 x^e] c_{\sigma^i}, \tag{3}$$

where $c_{\sigma^i} = \frac{1}{1 - (\sigma^i)^2}$. The pursuer-defendable sets are thus approximated by $\mathcal{Y}^i = \{q \in \mathbb{R}^2 \mid \|q - o^i\| \leq r^i\}$. Further, define $\mathcal{Y}^p := \cup_{i=1}^N \mathcal{Y}^i$ to be the joint pursuer-defendable set.

Remark 2: While in [2], an explicit defense manifold is constructed in terms of the Apollonius circles, any nonintersecting curve contained in $\mathcal{Y}^p$ with endpoints on opposite sides of $\partial \mathcal{R}^e_c$, as defined by crossing the major axis of the CRS, is a pursuer-defendable defense manifold. Hence, there exists a pursuer-defendable defense manifold provided that $\mathcal{Y}^p$ is connected and the Apollonius circles corresponding to the endpoint pursuers intersect $\partial \mathcal{R}^e_c$, as seen in Fig. 1.

Remark 3: To maximize a pursuer's likelihood to capture the evader, it is desirable to maximize the amount of a pursuer's

Apollonius circle contained in the CRS. For this reason, one approach is to attempt to directly control the center of a pursuer's Apollonius circle, o^i, using the mapping

$$\dot{x}^i = \tfrac{1}{c_{\sigma^i}}\dot{o}^i + (\sigma^i)^2 u^e, \tag{4}$$

where $\dot{o}^i$ are the desired dynamics for o^i.

IV. Equivalence between 1D Coverage Control and Consensus

A. Standard Coverage Control

Coverage control aims to address the problem of optimally distributing resources within a domain of interest. The mechanization of coverage control in this work is based on locational optimization and the definition of the locational cost [10]. The locational cost describes how well the configuration covers the domain and is given by

$$\mathcal{H}(p,t) := \sum_{i=1}^{N} \int_{\Omega^i(p,t)} \|q - p^i\|^2 \rho(q,t) dq, \tag{5}$$

where $p^i(t) \in \mathcal{M}(t) \subseteq \mathbb{R}^2$ is a mapping of the pursuers' positions onto the coverage domain, $\mathcal{M}(t)$, $\rho : \mathcal{M}(t) \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ is a time-varying density function assigning the importance of covering a given point in the domain, and $\Omega^i(p,t)$ forms a proper partition of the domain. In this work, a fixed density $\rho(q,t) = 1$ is assumed, but rather the domain of interest, $\mathcal{M}(t)$, is assumed to be time-varying [11]. From this definition of the locational cost, the centers of mass, $c^i(p,t) := \int_{\Omega^i(p,t)} q\rho(q,t)dq / \int_{\Omega^i(p,t)} \rho(q,t)dq$ are the only critical points of (5), and accordingly, the necessary position configuration for the agents to provide optimal coverage. Define c and p to be the stacked vectors of centers of mass and positions of the agents, respectively. The continuous-time version of Lloyd's algorithm, [12], can be shown to be a gradient descent strategy for minimizing (5):

$$\dot{p} = \kappa(c - p). \tag{6}$$

Consider a team of N heterogeneous pursuers where a weight, $w^i > 0$, accounts for how much more of the domain pursuer i can cover. Then the partition of choice, $\Omega^i(p,t)$, is the weighted Voronoi tessellation, where the weighted Voronoi cell for agent i is given by $\mathcal{V}^i(p,t) := \{q \in \mathcal{M}(t) \mid w^i\|q - p^i\| \leq w^j\|q - p^j\| \;\; \forall j \neq i\}$.

Further, suppose $\mathcal{M}(t)$, is represented by a nonintersecting curve, $\gamma : [0,L] \times \mathbb{R}_{\geq 0} \to \mathcal{M}(t)$, where L is the arclength of $\mathcal{M}(t)$. Thus, a correspondence between the mapped positions of the agents and the arclength of the curve can be obtained. With a slight abuse of notation, define $p^i \equiv q^i \in [0,L]$ such that $\gamma(q^i,t) = p^i(t)$. That is, treat p^i as both the distance along the arclength and its global position in $\mathbb{R}^2$. Similarly, let $\hat{r}^1(t) \equiv 0$, $\hat{r}^2(t) \equiv L$ be the endpoints of $\mathcal{M}(t)$. Through this parameterization, the corresponding centers of mass in $\mathcal{M}(t)$ are given in [13] in terms of the weights, w^i.

B. Implicitly Defined Coverage Domains in 2D Engagements

Using tools from graph theory, [14], we define the Laplacian matrix encoding the information exchange topology for the pursuers to demonstrate the equality between Lloyd's algorithm, (6), and leader-follower consensus dynamics.

Suppose, that each agent is able to exchange information with its immediate neighbors in the curve, $\gamma(q,t)$, i.e., the communication topology is given by a path graph. The adjacency matrix for the pursuers is given by $A \in \mathbb{R}^{N\times N}$, where $[A]_{ij} = \frac{1}{2}\frac{w^j}{w^i + w^j}$ if $j = i+1$ or $j = i-1$ and $[A]_{ij} = 0$ otherwise. Hence, A is tridiagonal. Further, define pursuer i's neighborhood set by $\mathcal{N}^i = \{j \in [N] \mid [A]_{ij} \neq 0\}$. Lloyd's algorithm, (6), can be written for each agent using adjacency information with external references, $\hat{r}(t) = [\hat{r}^1(t), \hat{r}^2(t)]^T$. Define the out-degree matrix to be the diagonal matrix $D_{\text{out}} \in \mathbb{R}^{N\times N}$ with $D_{\text{out}} = \text{diag}(A\mathbb{1}_N)$, where $\mathbb{1}_N \in \mathbb{R}^N$ is the vector of all 1's. We can define the Laplacian matrix associated to the system by $L = D_{\text{out}} - A$.

Lemma 4.1: Lloyd's algorithm can be expressed as

$$\dot{p} = \kappa\left(-L_f p + \tfrac{1}{2}B\hat{r}\right), \tag{7}$$

where $L_f = L + \frac{1}{2}B_r$, $B_r = \text{diag}(1,0,\ldots,0,1)$, and $B = \left[\begin{smallmatrix}1 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 1\end{smallmatrix}\right]^T$. Moreover, for two fixed reference points $\hat{r}^1, \hat{r}^2$, these dynamics asymptotically drive the agents to a unique equilibrium such that each agent lies in the convex hull of $\hat{r}^1, \hat{r}^2$, with spacing dictated by the weights, w^i.

Proof: The fact that it can be expressed in this form follows from the expressions for the c^i given in [13]. To see that L_f is positive definite, note that L and B_r are both positive semi-definite. Therefore, their sum is at least positive semi-definite. To show positive definiteness, it suffices to show that the intersection of their kernels is trivial. Because L is the Laplacian associated to a strongly connected digraph, $\ker(L) = \text{Span}\{\mathbb{1}_N\}$ and $\ker(B_r) = \text{Span}\{e_2, \ldots, e_{N-1}\}$, where e_i is the i^{th} canonical basis element of $\mathbb{R}^N$. Then $\ker(L) \cap \ker(B_r) = \{0\}$. Hence, L_f is positive definite and the unique equilibrium point for p is given by $p^* = \frac{1}{2}L_f^{-1}B\hat{r}$. For a single agent $i \neq 1, N$, its dynamics are

$$\dot{p^i} = \alpha^{i-1}p^{i-1} + \alpha^{i+1}p^{i+1} - (\alpha^{i-1} + \alpha^{i+1})p^i \implies$$
$$(p^i)^* = \tfrac{\alpha^{i-1}}{\alpha^{i-1}+\alpha^{i+1}}p^{i-1} + \tfrac{\alpha^{i+1}}{\alpha^{i-1}+\alpha^{i+1}}p^{i+1},$$

where $\alpha^{i-1} = \frac{1}{2}\frac{w^{i-1}}{w^i + w^{i-1}}, \alpha^{i+1} = \frac{1}{2}\frac{w^{i+1}}{w^i + w^{i+1}}$, and $(p^i)^*$ is the unique equilibrium point for p^i. So p^i is in the convex hull of its neighbors. Similarly, $(p^1)^* = \frac{\alpha^0}{\alpha^0 + \alpha^2}\hat{r}^1 + \frac{\alpha^2}{\alpha^0+\alpha^2}p^2$, where $\alpha^0 = \frac{1}{2}, \alpha^2 = \frac{1}{2}\frac{w^2}{w^1+w^2}$. So agent 1 is in the convex hull of $\hat{r}^1$ and p^2. A similar argument holds for agent N. Thus, all agents end up in the convex hull of $\hat{r}^1, \hat{r}^2$. ■

For time-varying reference points, $\hat{r}^1(t), \hat{r}^2(t)$, we show the boundedness of the error between the positions of the pursuers and the centers of mass, $c(p,t)$. To do so, we first present a few necessary intermediate results.

Lemma 4.2 (Similarity transform of L_f, [15]): Define $\ell_{i,j} := [L_f]_{i,j}$. Since $\ell_{i+1,i}\ell_{i,i+1} > 0$, $\forall i \in \{1,\ldots,N-1\}$,

L_f can be transformed into a symmetric tridiagonal matrix, J, given by $L_f = SJS^{-1}$, where

$$S := \mathrm{diag}(\delta_1, \dots, \delta_N),\ \delta_i := \begin{cases} 1, & \text{if } i = 1 \\ \sqrt{\frac{\prod_{j=1}^{i-1} \ell_{j+1,j}}{\prod_{j=1}^{i-1} \ell_{j,j+1}}}, & \text{otherwise.} \end{cases} \tag{8}$$

Lemma 4.3: The spectral condition number of L_f is defined to be $\chi(L_f) := \min\{\|U\|\|U^{-1}\| \in [1,\infty) \mid U\Lambda U^{-1} = L_f\}$, where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$ is the diagonal matrix of eigenvalues of L_f. Then

$$\chi(L_f) \le \sqrt{w_{\text{ratio}}} := \max_{(i,j)\in[N]\times[N]} \sqrt{\tfrac{w^i}{w^j}}. \tag{9}$$

Proof: By Lemma 4.2, we have $J = S^{-1}L_f S$, where J is a symmetric tridiagonal matrix. Because J is symmetric, it may be orthogonally diagonalized $J = U\Lambda U^T$ where $U \in \mathrm{O}(N)$ is orthogonal and $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_N)$. Hence, $L_f = SU\Lambda U^T S^{-1}$ and $\chi(L_f) \le \|SU\|\|U^T S^{-1}\| \le \|S\|\|U\|\|U^T\|\|S^{-1}\|$. However, because U is orthogonal, $\|U\|\|U^T\| = 1$. Thus,

$$\chi(L_f) \le \|S\|\|S^{-1}\| = \frac{\max_{i\in[N]} \delta_i}{\min_{i\in[N]} \delta_i} = \frac{\sqrt{w^1/\min_{j\in[N]} w^j}}{\sqrt{w^1/\max_{i\in[N]} w^i}} = \sqrt{w_{\text{ratio}}},$$

■

Corollary 4.4: If $w^1 = \ldots = w^N$, then $\chi(L_f) = 1$.

Theorem 4.5: Suppose $\sup_{t\in[t_0,t_f]} \|\dot{\hat{r}}(t)\| \le \overline{d}$. Then $\|p(t) - c(p(t),t)\|$ is bounded $\forall t \in [t_0, t_f]$.

Proof: Define the tracking error, $\xi := p - c = L_f p - \frac{1}{2}B\hat{r}$, with dynamics

$$\dot{\xi} = -\kappa L_f \xi - \tfrac{1}{2}B\dot{\hat{r}}. \tag{10}$$

Further, L_f's dominant eigenvalue is real and bounded by $0 < \lambda_1 \le \frac{1}{2}$, [16, Theorem 1]. Then by diagonalizablility of L_f, [17, Sec. 4.4], $\|\xi\|$ is bounded by

$$\|\xi(t)\| \le \sqrt{w_{\text{ratio}}}\|\xi(t_0)\| \exp(-\kappa\lambda_1(t-t_0)) + \tfrac{\overline{d}\sqrt{w_{\text{ratio}}}}{2\kappa\lambda_1}. \tag{11}$$

Thus, $\|p(t) - c(p(t),t)\|$ is bounded $\forall t \in [t_0, t_f]$. ■

It is important to note that the dynamics in (7) are consensus dynamics in a leader-follower network [18]. Each pursuer is a follower, running consensus dynamics while the reference points, $\hat{r}^1(t), \hat{r}^2(t)$, are the leaders of the network.

While it was assumed the dynamics in (7) were run with respect to the arclength parameterization of the agents, they may be run on the centers of the Apollonius circles, $o^i \in \mathbb{R}^2$, via the transformation in (4) as in the following:

$$\dot{o} = \kappa\left((-L_f \otimes I_2)o + \tfrac{1}{2}(B \otimes I_2)\hat{r}\right), \tag{12}$$

where $o = [o^{1^T}, \dots, o^{N^T}]^T$ and $\otimes$ is the Kronecker product. These are the dynamics in (7) run independently along each of the two dimensions. For static reference points $\hat{r}^1$ and $\hat{r}^2$, these dynamics converge to the same configuration as in (7), where $\mathcal{M}(t)$ is given by a straight line connecting the reference points, by Lemma 4.1. These dynamics can be interpreted as if $\mathcal{M}(t)$ is the manifold that linearly interpolates between the finite point sequence $\hat{r}^1(t), o^1(t), \dots, o^N(t), \hat{r}^2(t)$. Rather than defining a mapping to do coverage on an explicit defense manifold as in [2], the pursuers can be thought of as coordinating their efforts directly on $\mathcal{M}(t)$ and maintaining a pursuer-defendable defense manifold by using consensus dynamics.

V. Defense Manifold Maintenance via Control Barrier Functions

Although coverage control laws are given in [2] for pursuers to coordinate their efforts over a defense manifold, no guarantees are made as to the maintenance of its pursuer-defendability throughout the engagement. While in some games it cannot be guaranteed that pursuer-defendability is maintained for all time, the use of control barrier functions (CBFs), [6], enforces the maintenance of a pursuer-defendable defense manifold for as long as it is feasible.

A. CBF Definitions

The purpose of CBFs is to render a desired set forward invariant, as defined next. For our application, the desired set would be the set of configurations for which the defense manifold is maintained pursuer-defendable. We reproduce the definition of a time-varying CBF here for completeness. Define the safe set $\mathcal{C} := \{x \in D \mid h(x,t) \ge 0\} \subseteq \mathbb{R}^n$, where $h : D \subseteq \mathbb{R}^n \times \mathbb{R}_{\ge 0} \to \mathbb{R}$ is continuously differentiable in both arguments. The set $\mathcal{C}$ is forward invariant if for every $x_0 \in \mathcal{C}$, $x(t) \in \mathcal{C}$ for $x(t_0) = x_0$ and all $t \in [t_0, \infty)$.

Definition 3 (Time-Varying CBFs). Given a dynamical system $\dot{x} = f(x) + g(x)u$, the function $h(x,t)$ is a time-varying CBF if there exists a locally Lipschitz extended class $\mathcal{K}$ function, α, such that for all $x \in D$,

$$\sup_{u\in\mathcal{U}} [\tfrac{\partial h}{\partial t} + \mathcal{L}_f h(x,t) + \mathcal{L}_g h(x,t)u + \alpha(h(x,t))] \ge 0, \tag{13}$$

where $\mathcal{L}_f h(x,t), \mathcal{L}_g h(x,t)$ are the Lie derivatives of $h(x,t)$ along $f(x)$ and $g(x)$, respectively. From the condition in (13), define the set of control inputs $K(x,t) := \{u \in \mathcal{U} \mid \frac{\partial h}{\partial t} + \mathcal{L}_f h(x,t) + \mathcal{L}_g h(x,t)u + \alpha(h(x,t)) \ge 0\}$.

Lemma 5.1 (Forward-Invariance [8]): Given $\mathcal{C}$, if h is a time-varying CBF, then any Lipschitz continuous controller $u \in K(x,t)$ will render $\mathcal{C}$ forward invariant.

B. Maintenance of Pursuer-Defendable Defense Manifold

To ensure a pursuer-defendable defense manifold exists for all time throughout the engagement, we first assume that $\mathcal{Y}_p$ contains a pursuer-defendable manifold at time t_0. Then we would like to ensure $\mathcal{Y}^p$ is a connected set for all time. Intuitively, this means that for any two adjacent Apollonius circles, the distance between their centers must be less than or equal to the sum of their radii. This defines the following candidate CBF for maintaining connectivity between two neighboring pursuers:

$$h_{ij}(x^i, x^j, x^e) := r^i + r^j - \|o^i - o^j\|. \tag{14}$$

Define $x := [x^{1^T}, \dots, x^{N^T}, x^{e^T}]^T \in \mathbb{R}^{2N+2}$. Then the set of positions of the players such that connectivity is maintained between pursuers i and j is given by $\mathcal{C}_{ij} := \{x \in \mathbb{R}^{2N+2} \mid h_{ij}(x^i, x^j, x^e) \ge 0\}$. The set of configurations where $\mathcal{Y}^p$ is connected is given by $\mathcal{C}_{\text{conn}} := \cap_{(i,j)\in E} \mathcal{C}_{ij}$, where E is the edge set associated to the path graph topology.

While $\mathcal{C}_{\text{conn}}$ is the set of configurations where all neighboring Apollonius circles are connected, we also require that $\mathcal{Y}^1, \mathcal{Y}^N$ are intersecting with $\partial\mathcal{R}_c^e$. To resolve this, define

$$x^{i*}(x^i, x^e, t) := \underset{q\in\partial\mathcal{R}_c^e}{\text{argmin}}\, \|q - o^i\|, \quad i \in \{1, N\}.$$

Remark 4: Although x^{i*} may not, in general, be unique or continuous, provided x^i does not cross the major axis of the ellipse given by the CRS, x^{i*} can be picked such that it is varying continuously differentiably.

Then define the candidate time-varying CBFs

$$h_{i*}(x^i, x^{i*}, x^e) := r^i - \|o^i - x^{i*}\|, \tag{15}$$

for $i \in \{1, N\}$. Additionally define $\mathcal{C}_{\text{bd}} = \cap_{i\in\{1,N\}}\{x \in \mathbb{R}^{2N+2} \mid h_{i*}(x^i, x^e, t) \geq 0\}$. Thus, the set of configurations where a pursuer-defendable defense manifold exists is given by $\mathcal{C} := \mathcal{C}_{\text{conn}} \cap \mathcal{C}_{\text{bd}}$. This way, if each of the h_{ij}, h_{i*} are time-varying CBFs, $\mathcal{C}$ is forward invariant by Lemma 5.1.

Under the dynamics in (1) and $\alpha(z) = \tilde{\gamma}z^3$ for $\tilde{\gamma} > 0$, the conditions for h_{ij} and h_{i*} with respect to (13) are given by

$$\begin{aligned}&\tfrac{\partial h_{ij}}{\partial x^i}u^i + \tfrac{\partial h_{ij}}{\partial x^j}u^j + \tfrac{\partial h_{ij}}{\partial x^e}u^e + \tilde{\gamma}h_{ij}^3(x^i, x^j, x^e) \geq 0,\\ &\tfrac{\partial h_{i*}}{\partial x^i}u^i + \tfrac{\partial h_{i*}}{\partial x^e}u^e + \tfrac{\partial h_{i*}}{\partial x^{i*}}\left(\tfrac{\partial x^{i*}}{\partial x^i}u^i + \tfrac{\partial x^{i*}}{\partial x^e}u^e + \tfrac{\partial x^{i*}}{\partial t}\right)\\ &+ \tilde{\gamma}h_{i*}^3(x^i, x^{i*}, x^e) \geq 0.\end{aligned} \tag{16}$$

These constraints on $u = [u^{1^T}, \ldots, u^{N^T}]^T$ are inequality constraints, $A_{ij}u \leq b_{ij}$ and $A_{i*}u \leq b_{i*}$, where

$$\begin{aligned}A_{ij}^T &= [0, \ldots, 0, -\tfrac{\partial h_{ij}}{\partial x^i}, -\tfrac{\partial h_{ij}}{\partial x^j}, 0, \ldots, 0]^T \in \mathbb{R}^{2N}\\ b_{ij} &= \tfrac{\partial h_{ij}}{\partial x^e}u^e + \tilde{\gamma}h_{ij}^3(x^i, x^j, x^e),\\ A_{i*}^T &= [0, \ldots, 0, -\tfrac{\partial h_{i*}}{\partial x^i} - \tfrac{\partial h_{i*}}{\partial x^{i*}}\tfrac{\partial x^{i*}}{\partial x^i}, 0, \ldots, 0]^T \in \mathbb{R}^{2N}\\ b_{i*} &= \tfrac{\partial h_{i*}}{\partial x^e}u^e + \tfrac{\partial h_{i*}}{\partial x^{i*}}\left(\tfrac{\partial x^{i*}}{\partial x^e}u^e + \tfrac{\partial x^{i*}}{\partial t}\right) + \tilde{\gamma}h_{i*}^3(x^i, x^{i*}, x^e).\end{aligned}$$

If a Lipschitz continuous controller, u, satisfies these linear inequalities for all $t \in [t_0, t_f]$, a pursuer-defendable defense manifold exists for all time. To ensure a controller satisfies these inequalities, define the controller as the solution to the convex quadratically constrained quadratic program (QCQP)

$$\begin{aligned}u(x,t) = \underset{u\in\mathbb{R}^{2N}}{\text{argmin}}\, &\|u - \hat{u}(x,t)\|^2\\ \text{s.t.}\quad &A_{ij}u \leq b_{ij} \qquad \forall(i,j) \in E\\ &A_{i*}u \leq b_{i*} \qquad i \in \{1, N\}\\ &\|u^i\|^2 \leq (\overline{u}^i)^2 \qquad \forall i \in [N],\end{aligned} \tag{17}$$

where $\hat{u}(x,t)$ is a nominal control input. Intuitively, all pursuers follow a nominal control input, $\hat{u}$, until the connectivity of $\mathcal{Y}^p$ is about to be broken, at which point the QCQP modifies the nominal controller in a way that enforces the connectivity of $\mathcal{Y}^p$ while maintaining speed constraints.

C. Decentralized CBFs for Decentralized Capture Strategies

Solving the QCQP in (17) provides a centralized strategy because an inequality must be solved for all pursuers simultaneously. To decentralize the QCQP in (17), we follow a similar methodology that was applied in [7].

Theorem 5.2: If each agent has control input, u^i, which solves the following QCQP for all $t \in [t_0, t_f]$, the set $\mathcal{C}$ is forward invariant throughout the engagement and a pursuer-defendable defense manifold exists for all time $t \in [t_0, t_f]$:

$$\begin{aligned}u^i(x,t) = \underset{u^i\in\mathbb{R}^2}{\text{argmin}}\, &\|u^i - \hat{u}^i(x,t)\|^2\\ \text{s.t.}\quad &\tilde{A}_{ij}u^i \leq \tilde{b}_{ij} \qquad \forall j \in \mathcal{N}_i\\ &\tilde{A}_{i*}u^i \leq b_{i*} \qquad \text{if } i \in \{1, N\}\\ &\|u^i\|^2 \leq (\overline{u}^i)^2,\end{aligned} \tag{18}$$

where $\tilde{A}_{ij} = -\frac{\partial h_{ij}}{\partial x^i}, \tilde{b}_{ij} = \frac{\eta^i}{\eta^i+\eta^j}b_{ij}$, $\tilde{A}_{i*} = -\frac{\partial h_{i*}}{\partial x^i} - \frac{\partial h_{i*}}{\partial x^{i*}}\frac{\partial x^{i*}}{\partial x^i}$, and η^i, η^j are positive weights associated to pursuers i and j, respectively.

Proof: Suppose two neighboring pursuers i and j satisfy these constraints. By summing the inequalities:

$$-\tfrac{\partial h_{ij}}{\partial x^i}u^i - \tfrac{\partial h_{ij}}{\partial x^j}u^j \leq b_{ij} \implies A_{ij}u \leq b_{ij},$$

which is the same linear inequality in (17). Additionally, $\tilde{A}_{i*}u^i = A_{i*}u \leq b_{i*}$, so the other constraint is satisfied as well. Further, the control input, u^i, is Lipschitz by [19], so the control, u, is also Lipschitz, and by Lemma 5.1, the set $\mathcal{C}$ is forward invariant throughout the engagement. ∎

Remark 5: The weights, η^i, η^j correspond to a means by which the pursuers distribute b_{ij}, the weight of maintaining connectedness. A pursuer with larger η^i has a larger responsibility to maintain the connectedness between its neighbors.

Pursuers can run the decentralized consensus strategy presented in (12) for reference points $\hat{r}^1, \hat{r}^2$ on $\partial\mathcal{R}_c^e$ as their nominal control inputs.

VI. Numerical Simulations

Simulations were carried out in MATLAB 2020a. A smart evader was considered, using CBFs corresponding to avoiding pursuers and ensuring it has time to reach the goal set:

$$\begin{aligned}h_{ie}(x^i, x^e) &:= \|x^i - x^e\| - \bar{c}\varepsilon, \ \ \forall i \in [N]\\ h_{\text{goal}}(x^e, t) &:= [(t_f - t_0) - t]\overline{u}^e - \|x^e - c_{\mathcal{P}}\|,\end{aligned} \tag{19}$$

where $\bar{c} \geq 1$ ($\bar{c} = 1.05$ in this letter). To prioritize that the evader moves toward the goal, $c_{\mathcal{P}}$, the evader solves an analogous QCQP with objective function $(x^e - c_{\mathcal{P}})^T u^e$.

For performance comparison, an uncoordinated pure-pursuit strategy was considered where $u^i = \overline{u}^i\frac{x^e - x^i}{\|x^e - x^i\|}$ as well as a coverage strategy without the CBF.

$N \in \{2, 3, 4\}$ pursuers were considered for the strategies. 100 trials were performed for each team number and strategy combination, where the initial positions of the pursuers were uniformly sampled from the half-ellipse bounded by the minor axis and containing $c_{\mathcal{P}}$. This way, $\mathcal{Y}_p$ at time t_0 was guaranteed to satisfy the criteria to contain a pursuer-defendable defense manifold. $\overline{u}^i = 0.01$ [m/s], $\sigma^i = 0.5\ \forall i$. For the CBF and coverage strategies, $\dot{x}^i = \frac{1}{c_{\sigma^i}}\dot{o}^i + (\sigma^i)^2 u^e$, where $\dot{o}^i$ is given in (12) and the reference points $\hat{r}$ were chosen as follows. Define $\hat{n} = \left[\begin{smallmatrix}0 & -1\\ 1 & 0\end{smallmatrix}\right](c_{\mathcal{P}} - x^e)$. Then $\hat{r}^1$ was defined to be the point on $\partial\mathcal{R}_c^e$ such that it is on the line with slope $\hat{n}$ and the line passes through x^1. Similarly, $\hat{r}^2$ is the point such that the line passes through x^N. Further, $w^i = 1, \eta^i = r^i, \forall i \in [N]$,

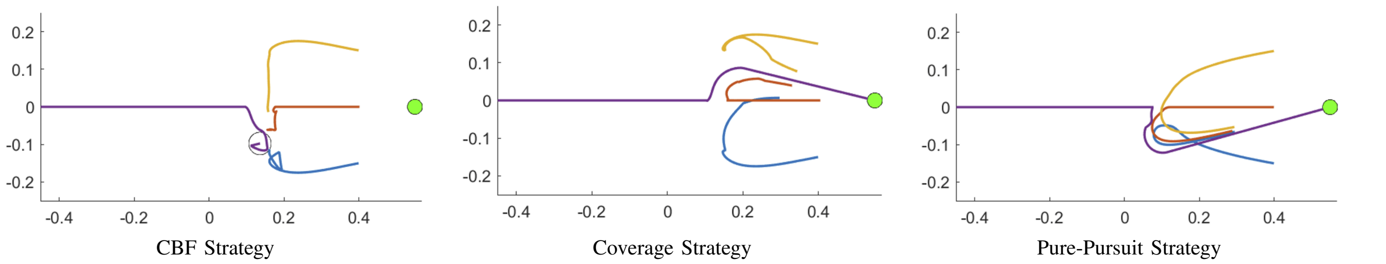

Fig. 2: Three example scenarios, all dimensions in meters. In the left scenario, the pursuers are employing the CBF strategy. In the middle, they are employing the coverage strategy, and in the right scenario, the pursuers are employing the pure-pursuit strategy. The evader's trajectory is displayed in purple and the pursuers' are displayed in yellow, orange, and blue. A circle is added for any agent whose distance to the evader is less than $\varepsilon = 0.03$ [m]. The trajectories are terminated either after capture is achieved or the evader made it to the target set, denoted by a green circle. Final time $t_f = 65$ [s].

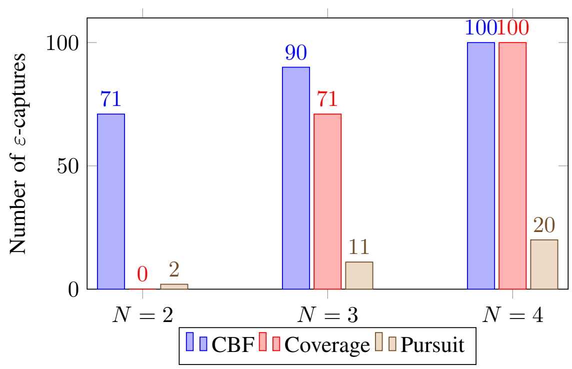

Fig. 3: Bar chart of results over 100 initial conditions with $N = 2, 3,$ and 4 pursuers. Significant improvement is seen in the CBF strategy compared to both pure-pursuit and coverage strategies.

$\tilde{\gamma} = 10^3, \kappa = 1$. The number of ε-captures was compared between the strategies. Trajectories are shown in Fig. 2 and results are shown in Fig. 3.

For $N = 2$ pursuers, only the CBF strategy was able to achieve ε-capture more than 50% of the time. For $N = 3$ pursuers, the CBF and coverage strategies were able to get ε-capture more than 50% of the time, but the CBF strategy was able to capture more than the coverage strategy alone. For $N = 4$ pursuers, the CBF and coverage strategies were able to capture every time. Overall, the CBF strategy performed better than both other strategies except for $N = 4$ pursuers, where the coverage strategy allowed for the pursuers to be packed closely enough together so that the evader could not get through the pursuers without allowing ε-capture.

VII. Conclusions

The problem of finite-time RA games with a faster evader is addressed. The proposed strategy is defense manifold maintenance via coverage control as in [2]. To guarantee a pursuer-defendable defense manifold exists throughout the engagement, CBFs are used to provide forward invariance of the set of configurations where a defense manifold remains pursuer-defendable. An equivalence between coverage control on one-dimensional manifolds and consensus in a leader-follower network is established and a bound on the error between the pursuers and the corresponding optimal coverage configuration is provided. Simulations demonstrate that this strategy is able to ε-capture the evader more frequently than the pure-pursuit and coverage strategies.

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